Lecture 03

Design of Doubly Reinforced Beam in Flexure

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- Mechanics of Doubly Reinforced Beam
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Learning Outcomes

- **At the end of this lecture, students will be able to;**
  - ***Define*** a doubly reinforced beam.
  - ***Understand*** the mechanics of a doubly reinforced beam.
  - ***Analyze*** and ***Design*** Doubly reinforced beam in flexure.

Background

- **Introduction**
  - As we know that flexural capacity of a beam is given by

$$\emptyset M_n = \emptyset A_s f_y \left(d - \frac{a}{2}\right)$$

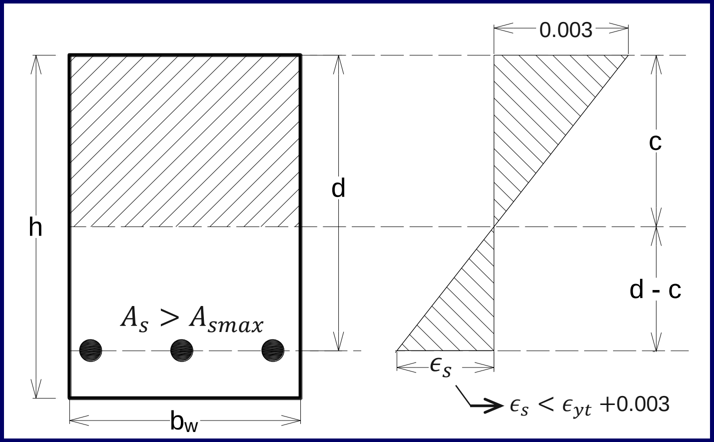
Taking $\emptyset M_n = M_u$

$$\emptyset A_s f_y \left(d - \frac{a}{2}\right) = M_u$$

  - This equation clearly shows that for a beam with specified material properties, we must increase either the area of steel A_s or the effective depth, d to meet the given demand moment M_u.
  - However, if the cross-sectional dimensions of beam are restricted due to architectural or some other considerations, then we are only left with increasing Area of steel.

Background

- **Introduction**
  - Unfortunately, we cannot increase the area of steel beyond the maximum reinforcement limit.
  - If A_s exceeds A_{smax}, the strain in concrete will reach a value of 0.003 before strain in the tension steel reaches $\epsilon_{ty} + 0.003$, thus violating the ACI Code recommendation for ensuring ductility.

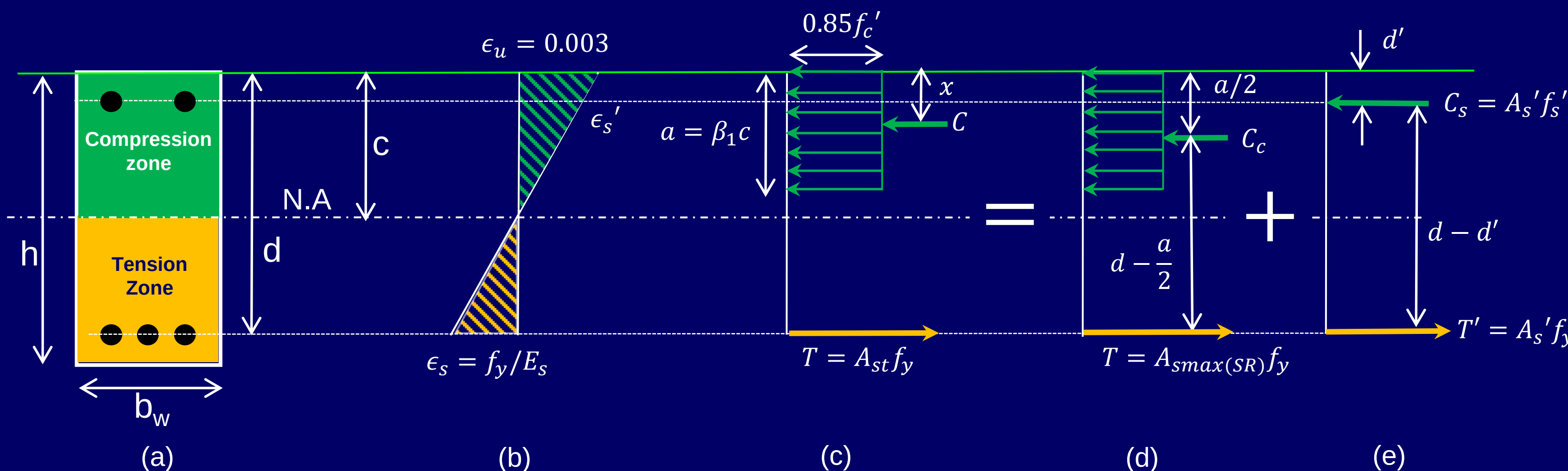


Background

- **Introduction**
  - However, if we strengthen the compression zone by either increasing f_c' or placing some additional reinforcement, the crushing of concrete will be delayed and the load at which strain reaches 0.003 will be increased.
  - When this happens, the range of A_{smax} is extended and A_s on the tension side can be raised without compromising ductility, which will also increase flexural capacity of the beam.
  - Practically this can be achieved simply by placing some additional steel on both faces (tension and compression) of the beam.
  - Such type of beam with both tension and compression reinforcement is known as Doubly reinforced beam.

Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**



The total nominal flexural capacity $M_{n(DR)}$ is given by;

$$M_{n(DR)} = M_{n1} + M_{n2}$$

M_{n1} = Maximum Resisting Moment provided by A_{smax}

M_{n2} = Maximum Resisting Moment provided by the Compression Steel

C_c = Compression force due to concrete in compression region.

C_s = Compression force in steel in compression region needed to balance the tension force in addition to the tension force provided by $A_{smax\,(SR)}$.

Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**

Design flexural capacity is given by

$$\varnothing M_{n(DR)} = \varnothing M_{n1} + \varnothing M_{n2}$$

Here,

$$\varnothing M_{n1} = \varnothing A_{s,max(SR)} \times f_y \left(d - \frac{a_{max}}{2}\right) \rightarrow lets\ call\ it\ \ \varnothing M_{n,max(SR)}$$

and

$$\varnothing M_{n2} = \varnothing A_s{}' f_s{}' \times (d - d')$$

So,

$$\varnothing M_{n(DR)} = \varnothing M_{n,max(SR)} + \varnothing A_s{}' f_s{}'(d - d')$$

Note:
For compression steel, we used $f_s{}'$ instead of f_y.
It is because we do not know whether the compression steel will yield or not.

Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**

For no failure, $\emptyset M_{n(DR)} \geq M_u$

Taking $M_u = \emptyset M_{n(DR)}$

$$M_u = \emptyset M_{n,max(SR)} + \emptyset A_s{}' f_s{}'(d - d')$$

Solving for $A_s{}'$ gives

$$A_s{}' = \frac{M_u - \emptyset M_{n,max(SR)}}{\emptyset f_s'(d - d')}$$

Taking $M_u - \emptyset M_{n,max(SR)} = M_u{}'$

$M_u{}'$ is the extra moment to be carried by compression steel

$$A_s{}' = \frac{M_u{}'}{\emptyset f_s'(d - d')} \quad \text{----- (3.1)}$$

What are the unknowns in eq. (3.1) ?

Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**

  - **Calculation of $\emptyset M_{n,max(SR)}$**

$$\emptyset M_{n,max(SR)} = 0.9 A_{s,max(SR)} f_y \left(d - \frac{a_{max}}{2}\right)$$

Here

$$a_{max} = \frac{A_{s,max(SR)} f_y}{0.85 f_c' b_w}$$

$$A_{smax(40)} = \frac{f_c' b_w d}{136}$$

$$A_{smax(60)} = \frac{f_c' b_w d}{223}$$

Putting value of $A_{s,max(SR)}$ and a_{max} and simplifying gives

$$\emptyset M_{n,max(SR),40} = 0.219 f_c' b_w d^2$$ ----- (3.2a)

and

$$\emptyset M_{n,max(SR),60} = 0.204 f_c' b_w d^2$$ ----- (3.2b)

For detailed derivation of eq.(3.2a) and (3.2b), refer to the Appendix.

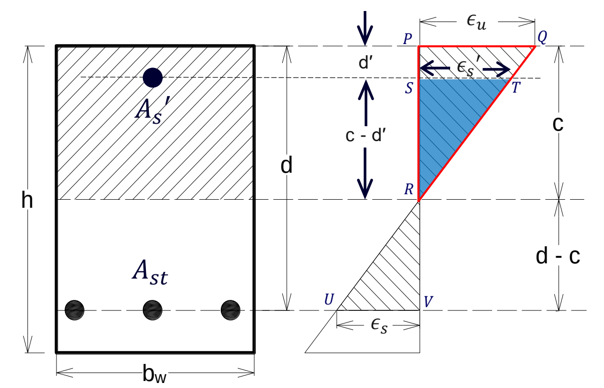
Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**

  - **Computation of f_s'**

$f_s' = E_s \epsilon_s'$

ϵ_s' can be calculated by similarity of triangles



From ΔPQR an ΔSTR , we have

$$\frac{ST}{SR} = \frac{PQ}{PR}$$

Since

$$ST = \epsilon_s', SR = c - d', PQ = \epsilon_u \ and \ \ PR = c$$

Therefore,

$$\frac{\epsilon_s'}{c - d'} = \frac{\epsilon_u}{c} \quad \Rightarrow \ \epsilon_s' = \frac{(c - d')\epsilon_u}{c} \quad \text{----- (3.3)}$$

What are the unknowns here?

Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**

  - **Computation of $f_s{}'$**

Again, from ΔPQR and ΔVUR, we have

$$\frac{PR}{PQ} = \frac{VR}{VU} \quad \Rightarrow \frac{c}{\epsilon_u} = \frac{d-c}{\epsilon_s}$$

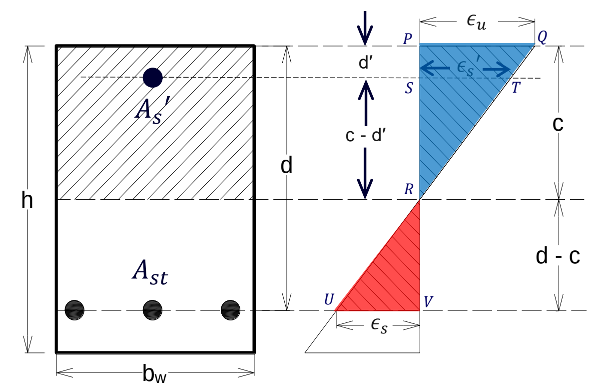
$$c = \left(\frac{\epsilon_u}{\epsilon_u + \epsilon_s}\right) d$$

$$c = \left(\frac{0.003}{0.003 + [\epsilon_{yt} + 0.003]}\right) d$$

Which on solving gives;

$$c = 0.4065d\ (f_y = 40ksi)\ \&$$

$$c = 0.3718d(f_y = 60ksi)$$



$\epsilon_u = 0.003$ (ACI 318 – 19, R21.2.2)

and

$\epsilon_s = \epsilon_{yt} + 0.003$ (ACI 318-19 Table 21.2.2)

Mechanics of Doubly Reinforced Beam

- **Determination of flexural capacity**

Computation of f_s'

- For Grade 40 steel: Putting $c = 0.4065d$ in eq. (3.3), we get

$$\epsilon_s' = \frac{(c - d')\epsilon_u}{c} = \frac{(0.4065d - d') \times 0.003}{0.4065d} = 0.003 - 0.00738d'/d$$

$$f_s' = E_s\epsilon_s' = 29000(0.003 - 0.00738d'/d)$$

$$f'_{s,40} = 87 - 214\frac{d'}{d} \le 40ksi$$ ----- (3.4a)

- For Grade 60 steel: Similarly, for $c = 0.3718d$

$$f_s' = E_s\epsilon_s' = 29000(0.003 - 0.008069d'/d)$$

$$f'_{s,60} = 87 - 234\frac{d'}{d} \le 60ksi$$ ----- (3.4b)

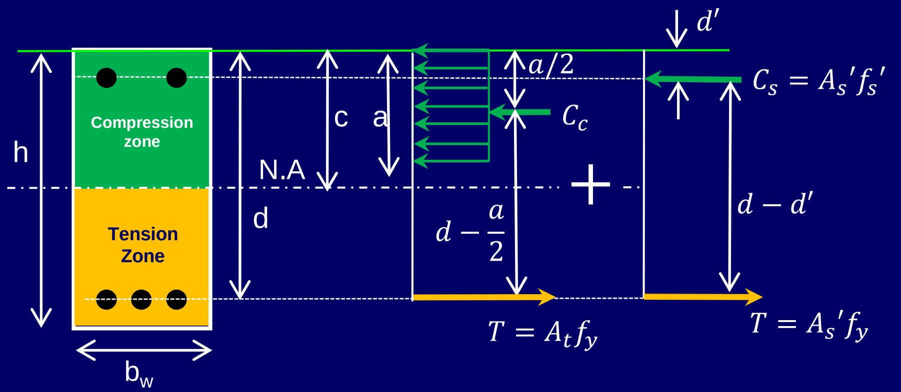
Important Note:

- Depending upon the ratio of d'/d, the steel in compression zone may or may not yields.
- It should be noted that yielding of compression steel is NOT necessary like that of tension steel.

Mechanics of Doubly Reinforced Beam

- **Determination of Maximum Reinforcement Limit**

Consider the figure shown



Equating horizontal forces;

$\sum F_x = 0$

$C_c + C_s = T$

$0.85f_c'ab_w + A_s'f_s' = A_{st}f_y$

When $a = \beta_1 c$ then $A_{st} = A_{s,max(DR)}$

$$A_{s,max(DR)}\ f_y = 0.85f_c'\beta_1(cb_w) + A_s'f_s'$$

$A_{s,max(DR)}$ is the maximum reinforcement limit for doubly reinforced beam

$$A_{s,max(DR)} = \frac{0.85f_c'\beta_1(cb_w)}{f_y} + \frac{A_s'f_s'}{f_y}$$

Mechanics of Doubly Reinforced Beam

- **Determination of Maximum Reinforcement Limit**

$$\frac{0.85 f_c' \beta_1 (c b_w)}{f_y} = A_{s,max(SR)}$$

$$A_{s,max(DR)} = A_{s,max(SR)} + \frac{f_s'}{f_y} (A_s')_{pvd}$$

$(A_s')_{pvd}$ is the steel area provided in the compression zone of the beam.

On putting value of $A_{s,max(SR)}$, we get

$$A_{s,max(DR),40} = \frac{f_c'}{136} b_w d + \frac{f_s'}{f_y} (A_s')_{pvd}$$ ----- (3.5a)

and

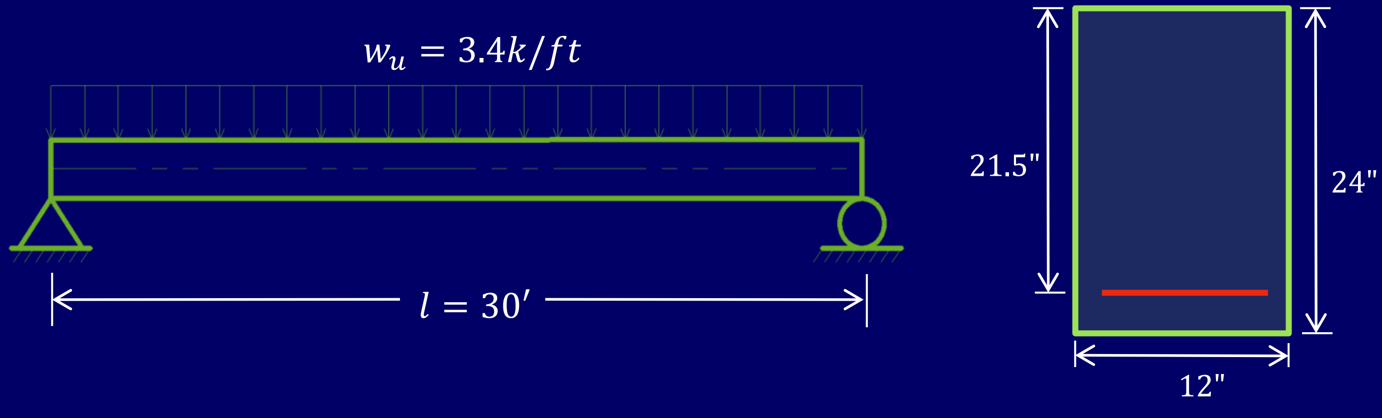
$$A_{s,max(DR),60} = \frac{f_c'}{223} b_w d + \frac{f_s'}{f_y} (A_s')_{pvd}$$ ----- (3.5b)

Design Procedure

- **Summary of design steps for a doubly reinforced beam**
  - Step No.1,2 and 3: Sizes, Loads and Analysis
  - Step No.4: Checking the behavior of section (whether SR or DR)
  - Step No.5: Determination of Compression steel area
  - Step No.6: Determination of Tension steel area
  - Step No.7: Detailing of reinforcement
  - Step No.8: Applying maximum reinforcement check
  - Step No.9: Drafting
  - Step No.10: Check for yielding of compression steel (optional)
  - Step No.11: Check flexural capacity of section (optional)

Design of Doubly Reinforced Beam in Flexure

- **Example 3.1**
  - Design the given reinforced concrete beam for an ultimate flexural demand of 4590 in-kip. The beam's cross-sectional dimensions are restricted. Material strengths to be used are also limited as $f_c' = 3ksi$ and $f_y = 40ksi$.



Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.1: Checking behavior of section**

Calculate area of steel based on singly reinforced beam mechanics

$$a = d - \sqrt{d^2 - \frac{2.614 M_u}{f_c' b_w}} = 21.5 - \sqrt{21.5^2 - \frac{2.614 \times 4590}{3 \times 12}} = 10.14"$$

$$A_s = \frac{M_u}{0.9 f_y \left(d - \frac{a}{2}\right)} = \frac{4590}{0.9 \times 40 \left(21.5 - \frac{10.14}{2}\right)} = 7.76 in^2$$

Now,

$$A_{s,max(SR),40} = \frac{f_c' b_w d}{136} = \frac{3 \times 12 \times 21.5}{136} = 5.69 in^2$$

Since $A_s > A_{s,max(SR)}$, the beam cannot be designed as singly reinforced. Now designing the beam as doubly reinforced.

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.2: Determination of Compression steel area**

$$M_u' = M_u - \emptyset M_{n,max(SR),40} = 4590 - 0.219 f_c' b_w d^2$$

$$M_u' = 4590 - 3644.38 = 945.62\ in.kip$$

Assuming $d' = 2.5"$

$$f_{s,40}' = 87 - 214\frac{d'}{d} = 87 - 214\frac{2.5}{21.5} = 62.12ksi > f_y$$

Use $f_{s,40}' = f_y = 40ksi$

Now,

$$A_s' = \frac{M_u'}{\emptyset f_s'(d - d')} = \frac{945.62}{0.9 \times 40(21.5 - 2.5)} = 1.38in^2$$

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.3: Determination of Tension steel area**

Tension steel area is the sum of maximum singly reinforcement and compression reinforcement.

$$A_{st} = A_{s,max(SR)} + A_s'$$

Here,

$$A_{s,max(SR)} = \frac{f_c' b_w d}{136} = 5.69 in^2$$

So, by substituting values, we get

$$A_{st} = 5.69 + 1.38$$

$$A_{st} = 7.07 in^2$$

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.4: Detailing of reinforcement**

Using #8 bar, with bar area $A_b = 0.79in^2$

No. of bars to be provided on tension side $= \frac{A_{st}}{A_b} = \frac{7.07}{0.79} = 8.95 \approx 9$

No. of bars to be provided on compression side $= \frac{A_s{}'}{A_b} = \frac{1.38}{0.79} = 1.8 \approx 2$

Hence, provide

- 9 #8 (in 3 layers) on tension side and
- 2 #8 (in 1 layer) on compression side.

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.5: Applying maximum reinforcement check**

Using eq. (3.5a)

$$A_{s,max(DR),40} = \frac{f_c'}{136} b_w d + \frac{f_s'}{f_y}(A_s')_{pvd}$$

$$A_{s,max(DR),40} = 5.69 + 1(2 \times 0.79) = 7.27 in^2$$

$f_s' = f_y$

- $(A_s')_{pvd}$ is area of provided compression steel.

Provided area of tension steel is,

$$A_{st,pvd} = 9(0.79) = 7.11 in^2$$

Since $A_{st,pvd} < A_{s,max(DR),40} \rightarrow \; OK$!

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.5: Applying maximum reinforcement check**
    - Although providing 9 bars in three layers is adequate, it is preferable to arrange as many bars as possible on the bottom most side to increase the effective depth "d".
    - Therefore, providing 10 #8 bars in three layers (4+4+2), we have

$$A_{st,pvd} = 10(0.79) = 7.9in^2 > A_{s,max(DR),40} \rightarrow Not\ OK!$$

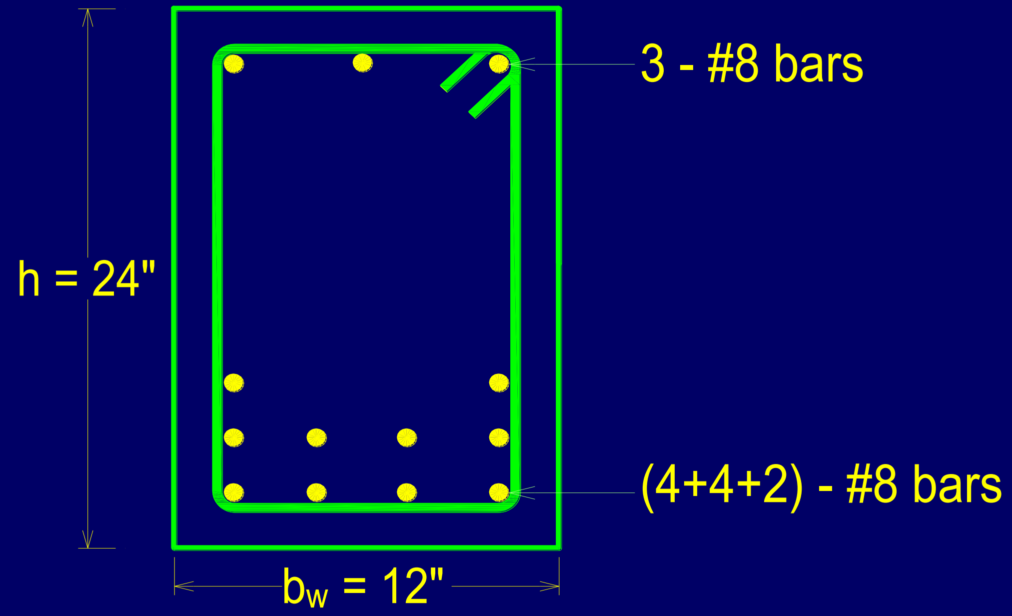
    - The range of $A_{s,max(DR),40}$ can be extended by simply providing a few extra bars in the compression zone only. Therefore, providing one extra #8 bar, we get

$$A_{s,max(DR),40} = 5.69 + (3 \times 0.79) = 8.06in^2 > A_{st,pvd} \rightarrow\ OK!$$

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.6: Drafting**

    Provide 10 #8 (in 3 layers) on tension side and 3 #8 (in 1 layer) on compression side of beam.



Design of Doubly Reinforced Beam in Flexure

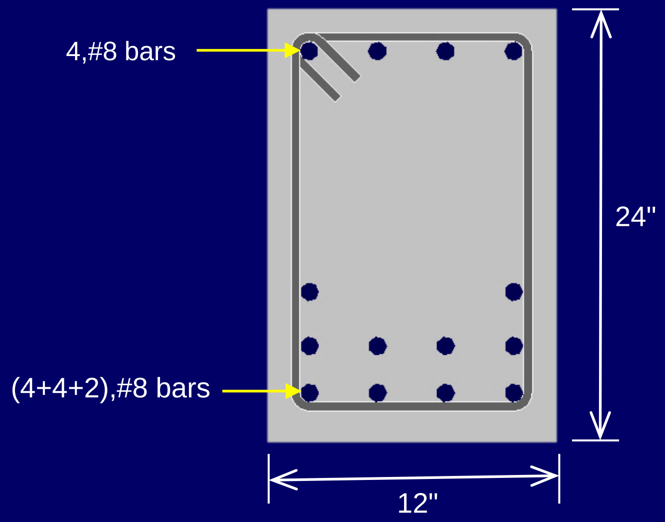
- **Solution**
  - **Step No.7: Check for yielding of compression steel (optional)**
    - Calculate actual d and d' as follows

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$$y_1 = 1.5 + \frac{3}{8} + \frac{8}{16} = 2.375"$$

$$y_2 = y_1 + \frac{8}{16} + 1.5 + \frac{8}{16} = 4.875"$$

$$y_3 = y_2 + \frac{8}{16} + 1.5 + \frac{8}{16} = 7.375"$$



Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.7: Check for yielding of compression steel (optional)**

$A_1 = A_2 = 4(0.79) = 3.16in^2$ and $A_3 = 2(0.79) = 1.58in^2$

Determine $\bar{y}$ as follows

$$\bar{y} = \frac{3.16(2.375) + 3.16(4.875) + 1.58(7.375)}{3.16 + 3.16 + 1.58} = 4.375"$$

$$d = h - \bar{y} = 24 - 4.375 = 19.625"$$

$$d' = 1.5 + \frac{3}{8} + \frac{8}{16} = 2.375"$$

Now

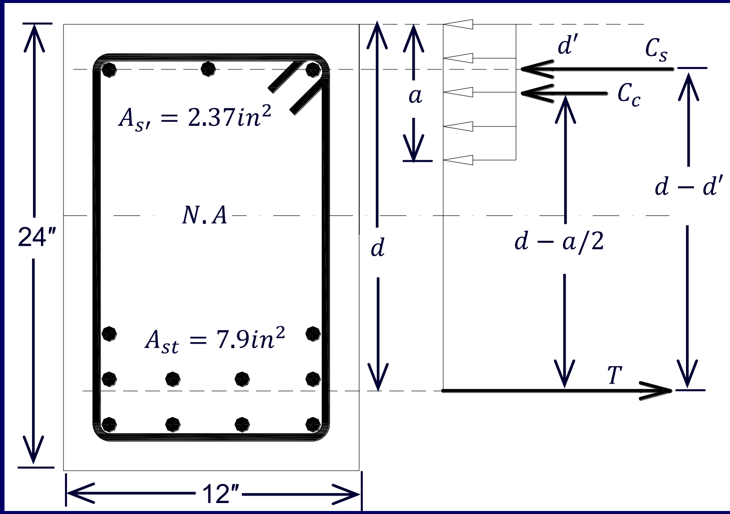
$$f'_{s,40} = 87 - 214\frac{2.375}{19.625} = 61.1ksi > 40ksi$$ → Compression steel yields!

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.8: Check flexural capacity of section (optional)**

Flexural capacity of doubly reinforced beam can be calculated as;

$$\emptyset M_{n(DR)} = \emptyset\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right]$$



Here,

$C_c = 0.85f_c'b_w a$ and $C_s = A_s'f_s'$

For doubly reinforced beam,

a is given by

$$a = \frac{A_{st}f_y - A_s'f_s'}{0.85f_c'b_w}$$

$$a = \frac{7.9 \times 40 - 2.37 \times 40}{0.85 \times 3 \times 12} = 7.23in$$

Design of Doubly Reinforced Beam in Flexure

- **Solution**
  - **Step No.8: Check flexural capacity of section (optional)**

Calculating compressive forces offered by concrete and compression steel

$C_c = 0.85 f_c' b_w a = 0.85 \times 3 \times 12 \times 7.23 = 221.238 kip$ and

$C_s = A_s' f_s' = 2.37(40) = 94.8 kip$

Now,

$$\emptyset M_{n(DR)} = 0.9\left[221.238\left(19.625 - \frac{7.23}{2}\right) + 94.8(19.625 - 2.375)\right]$$

$$\emptyset M_{n(DR)} = 4659.59\ in.k > M_u \rightarrow OK!$$

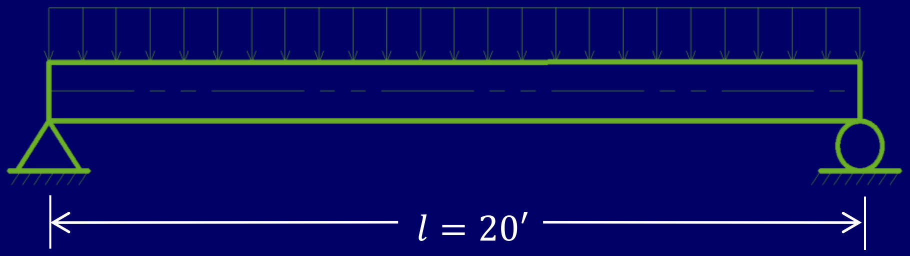
Design of Doubly Reinforced Beam in Flexure

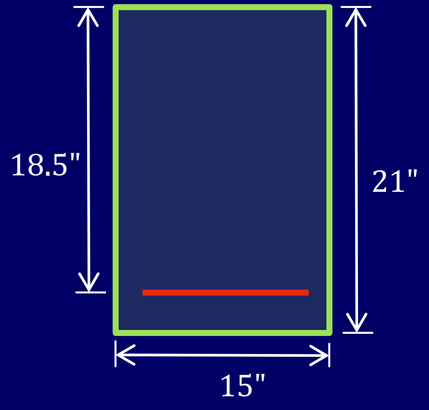
- **Example 3.2 (Class Activity)**

A simply supported beam is subjected to a factored flexural demand of 3130 in.kip. The beam's cross-sectional dimensions are restricted. The material strengths are also limited as $f_c' = 3ksi$ and $f_y = 60\ ksi$

a) State whether is it possible to design the beam as singly reinforced?

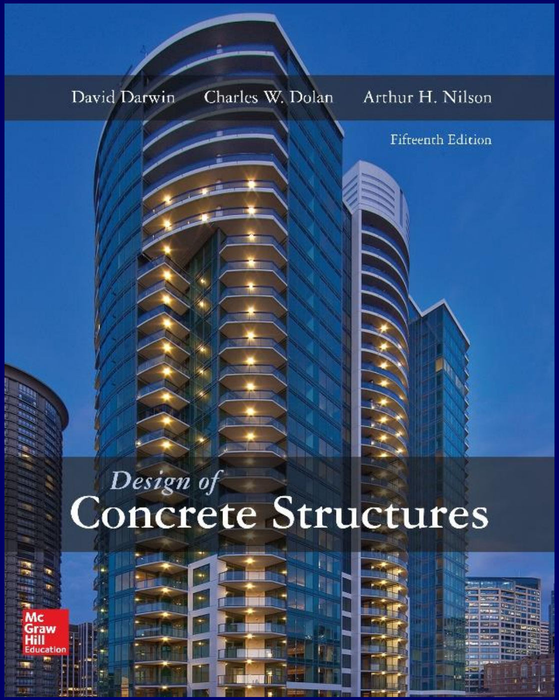
b) Design the section using #8 bar, and calculate the flexural capacity

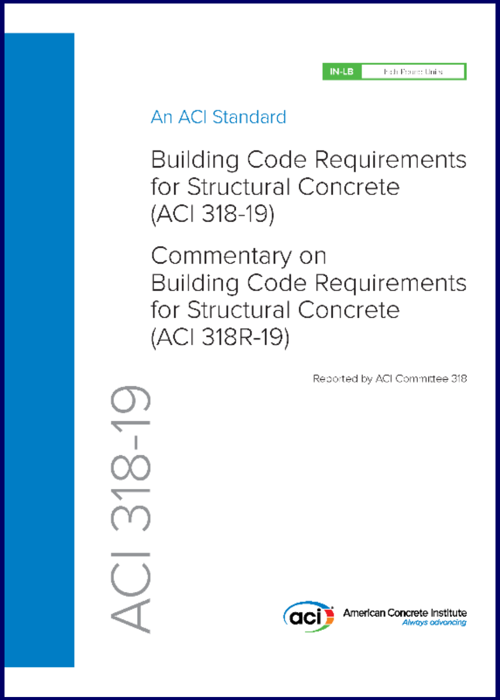




References

- Design of Concrete Structures 14th / 15th edition by Nilson, Darwin and Dolan.
- Building Code Requirements for Structural Concrete (ACI 318-19)





Appendix

- **Calculation of $\emptyset M_{n,max(SR)}$ for Grade 40 and Grade 60 steel**

As we know that

$$\emptyset M_{n,max(SR)} = 0.9A_{s,max(SR)}f_y\left(d - \frac{a_{max}}{2}\right) \text{ ----- (A)}$$

- **For Grade 40 steel:**

$$A_{s,max} = \frac{f_c' b_w d}{136} \quad and$$

$$a_{max} = \frac{A_{s,max}f_y}{0.85f_c' b_w} = \frac{f_c' b_w d/136 \times 40}{0.85f_c' b_w} = \frac{40}{136 \times 0.85}d$$

Putting these values in equation (A) , we get

$$\emptyset M_{n,max(SR),40} = 0.9\frac{f_c' b_w d}{136} \times 40\left(d - \frac{1}{2}\left[\frac{40}{136 \times 0.85}d\right]\right)$$

Appendix

- **Calculation of $\emptyset M_{n,max(SR)}$ for Grade 40 and Grade 60 steel**

  - **For Grade 40 steel:**

$$\emptyset M_{n,max(SR),40} = \frac{0.9 \times 40}{136} f_c' b_w d^2 - \frac{0.9 \times 40}{136} \times \frac{1}{2} \left[\frac{40}{136 \times 0.85}\right] f_c' b_w d^2$$

$$\emptyset M_{n,max(SR),40} = \frac{0.9 \times 40}{136} f_c' b_w d^2 - \frac{0.9 \times 40^2}{136^2 \times 0.85 \times 2} f_c' b_w d^2$$

$$\emptyset M_{n,max(SR),40} = \left[\frac{0.9 \times 40}{136} - \frac{0.9 \times 40^2}{136^2 \times 0.85 \times 2}\right] f_c' b_w d^2$$

Which on solving gives

$$\emptyset M_{n,max(SR),40} = 0.219 f_c' b_w d^2$$

Appendix

- **Calculation of $\emptyset M_{n,max(SR)}$ for Grade 40 and Grade 60 steel**

  - **For Grade 60 steel:**

Similarly,

$$A_{s,max,60} = \frac{f_c' b_w d}{223}$$

Putting in equation (A), will yield

$$\emptyset M_{n,max(SR),60} = \left[\frac{0.9 \times 60}{223} - \frac{0.9 \times 60^2}{223^2 \times 0.85 \times 2}\right] f_c' b_w d^2$$

Which on solving gives

$$\emptyset M_{n,max(SR),60} = 0.204 f_c' b_w d^2$$